JOURNAL OF MULTIVARIATE ANALYSIS 6, 295–308 (1976)

Equivalence of Measures for Some Class of Gaussian Random Fields

K. INOUE

Department of Mathematics, Shinshu University, Asahi-Machi, Matsumoto 390, Japan

Communicated by T. Hida

We consider two Gaussian measures P_1 and P_2 on $(C(G), \mathfrak{B})$ with zero expectations and covariance functions $R_1(x, y)$ and $R_2(x, y)$ respectively, where $R_\nu(x, y)$ is the Green's function of the Dirichlet problem for some uniformly strongly elliptic differential operator $A^{(\nu)}$ of order $2m$, $m \geqslant [d/2] + 1$, on a bounded domain G in $\mathbb{R}^d$ ($\nu = 1, 2$). It is shown that if the order of $A^{(2)} - A^{(1)}$ is at most $2m - [d/2] - 1$, then P_1 and P_2 are equivalent, while if the order is greater than $2m - [d/2] - 1$, then P_1 and P_2 are not always equivalent.

1. NOTATIONS

In this paper, we shall deal with equivalence of measures for some class of Gaussian random fields. In order to state our problem, we need the following notations. Let G be a bounded domain with a smooth boundary in the d-dimensional Euclidian space $\mathbb{R}^d$, and let m be a fixed integer such that $m \geqslant [d/2] + 1$. We denote by $C_0^\infty(G)$ the linear space of all infinitely differentiable functions on $\mathbb{R}^d$ with supports contained in G. Set

$$D^\alpha \varphi = \frac{\partial^{|\alpha|}\varphi(x)}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}},$$

where $x = (x_1, ..., x_d) \in \mathbb{R}^d$, $\alpha = (\alpha_1, ..., \alpha_d)$, $|\alpha| = \sum_{j=1}^d \alpha_j$ and α_j's are non-negative integers. The Sobolev space $H^n(G)$, $n \geqslant 0$, is a powerful tool in our discussion. The inner product and the norm in $H^n(G)$ are given by

$$(\varphi, \psi)_n = \sum_{|\alpha| \leqslant n} \int_G D^\alpha\varphi(x) \cdot \overline{D^\alpha\psi(x)}\, dx,$$

Received January 13, 1975; revised July 15, 1975.

AMS 1970 Subject classifications: Primary 60G30; secondary 35J40, 60G15, 60J99.

Key words and phrases: Equivalence of Gaussian measures, Green's function of the Dirichlet problem.

and $\| \varphi \|_n = (\varphi, \varphi)_n^{1/2}$, respectively. Denote by $H_0^n(G)$ the closure of $C_0^\infty(G)$ in $H^n(G)$. Let us consider a uniformly strongly elliptic differential operator A of order $2m$ expressed in the form

$$A\varphi = \sum_{|\alpha|,|\beta| \leqslant m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x) \cdot D^\beta \varphi(x)),$$

where $a_{\alpha\beta}(x)$ are infinitely differentiable real bounded functions on G such that $a_{\alpha\beta}(x) = a_{\beta\alpha}(x)$ ($| \alpha |, | \beta | \leqslant m$). Then we shall say that A belongs to the class $\mathfrak{A}_m$ if there exists a positive constant C such that

$$(\varphi, A\varphi)_0 \geqslant C \cdot \| \varphi \|_m^2 \qquad \text{for any} \quad \varphi \in C_0^\infty(G).$$

Now, let us introduce a Gaussian measure on a measurable space $(C(G), \mathfrak{B})$, where $C(G)$ is the set of all real continuous functions $\xi(x)$ on G and $\mathfrak{B}$ is the σ-algebra generated by cylinder subsets of $C(G)$. Let $R(x, y)$ be the Green's function of the Dirichlet problem for an operator $A \in \mathfrak{A}_m$ on the domain G in the following sense: for each $y \in G$,

$$R(\cdot, y) \in H_0^m(G) \qquad \text{and} \qquad AR(\cdot, y) = \delta(\cdot - y),$$

where $\delta(x)$ is the so-called delta function. It must be pointed out that $R(x, y)$ is a real continuous positive-definite function on $G \times G$ and is Hölder continuous in each argument (see Section 3). Therefore we can prove, by using the technique of Fernique [3], that there uniquely exists a probability measure P on $(C(G), \mathfrak{B})$, for which $\{\xi(x), x \in G\}$ is a Gaussian random field with zero expectation and covariance function $R(x, y)$. The set of all probability measures on $(C(G), \mathfrak{B})$ obtained in this way from operators in $\mathfrak{A}_m$ will be denoted by $\mathfrak{P}_m$. With these notations we are now ready to state our problem in the following section.

2. Statement of Results

It is well known that any two Gaussian measures P_1 and P_2 on $(C(G), \mathfrak{B})$ are either equivalent ($P_1 \sim P_2$) or mutually singular ($P_1 \perp P_2$). We consider the particular case, where P_ν is the probability measure in $\mathfrak{P}_m$ determined by an operator $A^{(\nu)} \in \mathfrak{A}_m$ ($\nu = 1, 2$). We shall give a condition for P_1 and P_2 to be equivalent in terms of order$(A^{(2)} - A^{(1)})$ which is the order of the operator $A^{(2)} - A^{(1)}$.

Let $\mathfrak{A}_m{}^1$ be the set of all operators $A \in \mathfrak{A}_m$ of the form

$$A = \sum_{p=1}^{N} L^{(p)*} L^{(p)}, \tag{2.1}$$

where N is a positive integer, $L^{(p)}$ is a differential operator of order at most m with real bounded continuous coefficients, and $L^{(p)*}$ is the differential operator formally adjoint to $L^{(p)}$ $(1 \leqslant p \leqslant N)$. Let $\mathfrak{A}_m^2$ be the set of all operators A of order $2m$ defined by

$$A = \sum_{k=0}^{m} C_k(-\Delta)^{m-k}, \qquad \Delta = \sum_{j=1}^{d} \partial^2/\partial x_j^2, \tag{2.2}$$

where C_k's are nonnegative constants. It is easy to see the relation $\mathfrak{A}_m^2 \subset \mathfrak{A}_m^1 \subset \mathfrak{A}_m$. Now our results are stated as follows.

THEOREM 1. *Let* $A^{(1)}$ *and* $A^{(2)}$ *be in* $\mathfrak{A}_m^1$ *and* $\mathfrak{A}_m$, *respectively. If* $\operatorname{order}(A^{(2)} - A^{(1)}) \leqslant 2m - [d/2] - 1$, *then the measures* P_1 *and* P_2 *are equivalent.*

THEOREM 2. *Let* $A^{(\nu)}$ *be in* $\mathfrak{A}_m^2$ $(\nu = 1, 2)$. *The measures* P_1 *and* P_2 *are equivalent if and only if* $\operatorname{order}(A^{(2)} - A^{(1)}) \leqslant 2m - [d/2] - 1$.

3. Green's Function and Reproducing Kernel Hilbert Space

In this section, we summarize some results on the Green's function of the Dirichlet problem for an operator $A \in \mathfrak{A}_m$. Let us consider the equation

$$u \in H_0^m(G), \qquad Au = f, \tag{3.1}$$

where f is a given element of $H_0^{m*}(G)$ the dual space of $H_0^m(G)$. Then there exists a continuous linear operator R from $H_0^{m*}(G)$ to $H_0^m(G)$ such that

$$ARf = f \tag{3.2}$$

holds for any $f \in H_0^{m*}(G)$ ([1]). If we restrict the domain of the operator R to the space $L^2(G) = H^0(G)$, then we have a continuous linear operator R from $L^2(G)$ to $L^2(G)$, where the image of $L^2(G)$ under R is contained in $H^{2m}(G)$ ([1]). The kernel theorem shows the operator R can be written in the form

$$(Rf)(x) = \int_G R(x, y) f(y)\, dy, \qquad f \in L^2(G), \tag{3.3}$$

where $R(x, y)$ is a bounded continuous function on $G \times G$ and $R(x, y) \in H^m(G \times G)$ ([1]). Further we can prove that $R(x, y)$ is a real positive-definite function on $G \times G$ and is Hölder continuous in each argument. The operator R and the function $R(x, y)$ are called the *Green operator* and the *Green's function*, respectively, of the Dirichlet problem for the operator A on the domain G.

We will now proceed to the reproducing kernel Hilbert space (RKHS) arising from the Green's function $R(x, y)$. First, introduce an inner product on $C_0^\infty(G)$ defined by

$$\langle \varphi, \psi \rangle = (\varphi, A\psi)_0, \qquad \varphi, \psi \in C_0^\infty(G). \tag{3.4}$$

Set $|\varphi| = \langle \varphi, \varphi \rangle^{1/2}$ and denote by $H(A)$ the Hilbert space defined by the completion of $C_0^\infty(G)$ with respect to this norm. Two norms $|\varphi|$ and $\|\varphi\|_m$, that were given before, give equivalent topologies on $C_0^\infty(G)$, so we can identify the set $H(A)$ with the set $H_0^m(G)$. It is noted that $H(A)$ is a separable Hilbert space consisting of continuous functions on G. Further we see the following

LEMMA 1. *The space $H(A)$ is a* RKHS *with the reproducing kernel $R(x, y)$, $x, y \in G$. That is, $H(A)$ is the smallest Hilbert space satisfying the following two conditions*:

(1) $R(\cdot, y) \in H(A)$ *for any* $y \in G$,

(2) $\langle u(\cdot), R(\cdot, y) \rangle = u(y)$ *for any* $u \in H(A)$ *and any* $y \in G$.

4. THE DIRECT PRODUCT OF A REPRODUCING KERNEL HILBERT SPACE

The notion of the direct product of a RKHS ([2]) plays an important role in our discussion. Let $H(R)$ be an arbitrary RKHS with a reproducing kernel $R(x, y)$, which is a real-valued function on $G \times G$. We assume that $H(R)$ is a separable Hilbert space. The direct product $H^{\otimes}(R)$ of $H(R)$ is the RKHS with a reproducing kernel $(R \otimes R)[(x^1, y^1), (x^2, y^2)]$, $(x^1, y^1), (x^2, y^2) \in G \times G$, defined by

$$(R \otimes R)[(x^1, y^1), (x^2, y^2)] = R(x^1, x^2) \cdot R(y^1, y^2).$$

We denote by $\langle u, v \rangle$ and $\langle\!\langle f, g \rangle\!\rangle$ the inner products in $H(R)$ and $H^{\otimes}(R)$ respectively. Set $|u| = \langle u, u \rangle^{1/2}$ and $\|f\| = \langle\!\langle f, f \rangle\!\rangle^{1/2}$. The space $H^{\otimes}(R)$ is to be a set of functions on $G \times G$. Especially, $H^{\otimes}(R)$ contains functions $u \otimes v$ on $G \times G$ given by $(u \otimes v)(x, y) = u(x) \cdot v(y)$, where $u, v \in H(R)$. For any $u, v, u', v' \in H(R)$, we have

$$\langle\!\langle u \otimes v, u' \otimes v' \rangle\!\rangle = \langle u, u' \rangle \cdot \langle v, v' \rangle. \tag{4.1}$$

Suppose $\{v_n\}_{n=1,2,\ldots}$ is a complete orthonormal system in $H(R)$. Then we see that the double sequence $\{v_k \otimes v_l\}_{k,l=1,2,\ldots}$ is the complete orthonormal system in $H^{\otimes}(R)$ and further that the space $H^{\otimes}(R)$ consists of all functions f on $G \times G$ of the form

$$f(x, y) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \alpha_{kl} \cdot (v_k \otimes v_l)(x, y) \tag{4.2}$$

with

$$\|f\|^2 = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |\alpha_{kl}|^2 < \infty. \tag{4.3}$$

It is easily seen that the series (4.2) converges absolutely for each $x, y \in G$.

LEMMA 2. *For any $f \in H^{\otimes}(R)$ and any $x, y \in G$, we have*

(1) $f(x, \cdot), f(\cdot, y) \in H(R)$,

(2) $\|f(x, \cdot)\|^2 \leqslant R(x, x) \cdot \|f\|^2$, $\quad \|f(\cdot, y)\|^2 \leqslant R(y, y) \cdot \|f\|^2$.

Proof. Let f be represented in the form (4.2) and let x be any fixed point in G. We have, in view of (4.3),

$$\begin{aligned}
\sum_{l=1}^{\infty} \left| \sum_{k=1}^{\infty} \alpha_{kl} v_k(x) \right|^2 &\leqslant \sum_{l=1}^{\infty} \left(\sum_{k=1}^{\infty} |\alpha_{kl}|^2 \right) \left(\sum_{k=1}^{\infty} |v_k(x)|^2 \right) \\
&= \left(\sum_{k=1}^{\infty} |v_k(x)|^2 \right) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |\alpha_{kl}|^2 \\
&= \left(\sum_{k=1}^{\infty} |\langle v_k(\cdot), R(\cdot, x)\rangle|^2 \right) \cdot \|f\|^2 \\
&= \| R(\cdot, x)\|^2 \cdot \|f\|^2 = R(x, x) \cdot \|f\|^2 < \infty,
\end{aligned}$$

which implies the series $\sum_{l=1}^{\infty} (\sum_{k=1}^{\infty} \alpha_{kl} v_k(x)) \cdot v_l(\cdot)$ converges strongly in $H(R)$ to some element $g_x(\cdot) \in H(R)$. This involves, for each $y \in G$,

$$f(x, y) = \sum_{l=1}^{\infty} \left(\sum_{k=1}^{\infty} \alpha_{kl} v_k(x) \right) v_l(y) = g_x(y),$$

which proves $f(x, \cdot) \in H(R)$. Further we obtain

$$\begin{aligned}
\|f(x, \cdot)\|^2 &= \|g_x\|^2 \\
&= \sum_{l=1}^{\infty} \left| \sum_{k=1}^{\infty} \alpha_{kl} v_k(x) \right|^2 \leqslant R(x, x) \cdot \|f\|^2.
\end{aligned}$$

The rest of the lemma can be proved analogously.

LEMMA 3. *For any* $f \in H^{\otimes}(R)$ *and any* $u \in H(R)$, *a function* f_u *on* G *given by* $f_u(x) = \langle f(x, \cdot), u(\cdot)\rangle$ *satisfies*

(1) $f_u \in H(R)$,

(2) $|f_u| \leqslant \|f\| \cdot |u|$.

Proof. Let f be represented in the form (4.2) and let x be any fixed point in G. We have shown in the proof of Lemma 2 that

$$f(x, \cdot) = \sum_{l=1}^{\infty} \left(\sum_{k=1}^{\infty} \alpha_{kl} v_k(x) \right) \cdot v_l(\cdot) \qquad \text{in} \quad H(R).$$

Then,

$$\begin{aligned} f_u(x) &= \langle f(x, \cdot), u(\cdot)\rangle \\ &= \sum_{l=1}^{\infty} \left(\sum_{k=1}^{\infty} \alpha_{kl} v_k(x) \right) \langle v_l , u\rangle \\ &= \sum_{k=1}^{\infty} \left(\sum_{l=1}^{\infty} \alpha_{kl} \langle v_l , u\rangle \right) v_k(x), \end{aligned}$$

since

$$\begin{aligned} \left(\sum_{l=1}^{\infty} \sum_{k=1}^{\infty} | \alpha_{kl} v_k(x)| \cdot |\langle v_l , u\rangle| \right)^2 &\leqslant \left\{ \sum_{l=1}^{\infty} \left(\sum_{k=1}^{\infty} | \alpha_{kl} v_k(x)| \right)^2 \right\} \left\{ \sum_{l=1}^{\infty} |\langle v_l , u\rangle|^2 \right\} \\ &\leqslant R(x, x) \cdot \|f\|^2 \cdot |u|^2 < \infty. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \left| \sum_{l=1}^{\infty} \alpha_{kl} \langle v_l , u\rangle \right|^2 &\leqslant \sum_{k=1}^{\infty} \left(\sum_{l=1}^{\infty} | \alpha_{kl} |^2 \right) \left(\sum_{l=1}^{\infty} |\langle v_l , u\rangle|^2 \right) \\ &= \|f\|^2 \cdot |u|^2 < \infty, \end{aligned}$$

which implies the series $\sum_{k=1}^{\infty} (\sum_{l=1}^{\infty} \alpha_{kl} \langle v_l , u\rangle) \, v_k(\cdot)$ converges strongly in $H(R)$ to some element $g(\cdot) \in H(R)$. It follows that $f_u(x) = g(x)$ and consequently $f_u \in H(R)$. We then obtain

$$|f_u|^2 = |g|^2 = \sum_{k=1}^{\infty} \left| \sum_{l=1}^{\infty} \alpha_{kl} \langle v_l , u\rangle \right|^2 \leqslant \|f\|^2 \cdot |u|^2,$$

which completes the proof.

In the rest of this section, we assume that $R(x, y)$ is the Green's function corresponding to an operator $A \in \mathfrak{A}_m$. We have shown in Lemma 1 that $R(x, y)$ is the reproducing kernel of the Hilbert space $H(A)$. For this reason, it is convenient to denote by $H^{\otimes}(A)$ the direct product of $H(A)$.

LEMMA 4. *Let $A \in \mathfrak{A}_m{}^1$ be represented in the form (2.1). Then the inner products in $H(A)$ and $H^{\otimes}(A)$ can be represented as follows;*

$$\langle u, v\rangle = \sum_{p=1}^{N} (L^{(p)}u, L^{(p)}v)_0\,, \qquad u, v \in H(A), \tag{4.4}$$

$$\langle\!\langle f, g\rangle\!\rangle = \sum_{p=1}^{N}\sum_{q=1}^{N} (L_x^{(p)}L_y^{(q)}f(x, y), L_x^{(p)}L_y^{(q)}g(x, y))_{L^2(G\times G)}\,, \qquad f, g \in H^{\otimes}(A), \tag{4.5}$$

where $(F_1, F_2)_{L^2(G\times G)}$ denotes the inner product in the space $L^2(G \times G)$ of square integrable functions on $G \times G$.

Proof. We can easily show (4.4) by (3.4). In order to prove (4.5), we have only to show, for any $f \in H^{\otimes}(A)$,

$$\|f\|^2 = \sum_{p=1}^{N}\sum_{q=1}^{N} \|L_x^{(p)}L_y^{(q)}f(x, y)\|^2_{L^2(G\times G)}\,, \tag{4.6}$$

where $\|F\|_{L^2(G\times G)} = (F, F)^{1/2}_{L^2(G\times G)}$. We denote by $\mathfrak{H}$ the subset of $H^{\otimes}(A)$ consisting of all functions f on $G \times G$ of the form

$$f(x, y) = \sum_{k=1}^{n_1}\sum_{l=1}^{n_2} \alpha_{kl}\cdot(v_k \otimes v_l)(x, y),$$

where n_j's are positive integers. Then $\mathfrak{H}$ is a dense linear subset of $H^{\otimes}(A)$. Using (4.1) and (4.4), it is easy to prove (4.6) for any $f \in \mathfrak{H}$. We will now proceed to the general case. For any $f \in H^{\otimes}(A)$, there exist $f_n \in \mathfrak{H}$, $n = 1, 2,\ldots$, such that $\lim_{n\to\infty} \|f_n - f\| = 0$. We denote by $F_n^{p,q}$ the function in $L^2(G \times G)$ given by

$$F_n^{p,q}(x, y) = L_x^{(p)}L_y^{(q)}f_n(x, y), \qquad 1 \leqslant p, q \leqslant N, \quad n = 1, 2,\ldots .$$

By using the relation (4.6) for each f_n, we can prove that $\{F_n^{p,q}\}_{n=1,2,\ldots}$ is a Cauchy sequence in $L^2(G \times G)$ and it converges to some element

$F_\infty^{p,q} \in L^2(G \times G)$, which is determined uniquely by f $(1 \leqslant p, q \leqslant N)$. We can then show the following relations:

$$\begin{cases} \|f\|^2 = \sum_{p=1}^{N} \sum_{q=1}^{N} \|F_\infty^{p,q}\|^2_{L^2(G\times G)}, \\ F_\infty^{p,q}(x,y) = L_x^{(p)} L_y^{(q)} f(x,y), \qquad 1 \leqslant p, q \leqslant N, \end{cases}$$

which proves the relation (4.6) for any $f \in H^\otimes(A)$.

LEMMA 5. *Let $A \in \mathfrak{A}_m{}^1$ be represented in the form* (2.1). *The space $H^\otimes(A)$ consists of all continuous functions f on $G \times G$ satisfying the following three conditions*:

(1) $f(x, \cdot), f(\cdot, y) \in H_0{}^m(G)$ *for any* $x, y \in G$,

(2) $(f(x, \cdot), A\varphi(\cdot))_0 \in H_0{}^m(G)$ *for any* $\varphi \in C_0{}^\infty(G)$,

(3) $\sum_{p=1}^{N} \sum_{q=1}^{N} \int_G \int_G |L_x^{(p)} L_y^{(q)} f(x,y)|^2 \, dx \, dy < \infty.$

Proof. If $f \in H^\otimes(A)$, then Lemmas 2, 3, and 4 guarantee the conditions (1), (2), and (3) respectively. The continuity of f follows from the one of the reproducing kernel of $H^\otimes(A)$. Conversely, let f be a continuous function on $G \times G$ satisfying the above three conditions. We have, for any $g \in H^\otimes(A)$,

$$\left| \sum_{p=1}^{N} \sum_{q=1}^{N} (L_x^{(p)} L_y^{(q)} g(x,y), L_x^{(p)} L_y^{(q)} f(x,y))_{L^2(G\times G)} \right| \leqslant C_f \cdot \|g\|,$$

where

$$C_f = \left\{ \sum_{p=1}^{N} \sum_{q=1}^{N} \|L_x^{(p)} L_y^{(q)} f(x,y)\|^2_{L^2(G\times G)} \right\}^{1/2}.$$

Therefore, by the theorem of Riesz, there exists a function $f_0 \in H^\otimes(A)$ such that, for any $g \in H^\otimes(A)$,

$$\sum_{p=1}^{N} \sum_{q=1}^{N} (L_x^{(p)} L_y^{(q)} g(x,y), L_x^{(p)} L_y^{(q)} f(x,y))_{L^2(G\times G)} = \langle\!\langle g, f_0 \rangle\!\rangle.$$

If we take now a function $g \in H^{\otimes}(A)$ given by $g(x, y) = \varphi(x) \cdot \psi(y)$, where φ and ψ are arbitrary functions in $C_0^{\infty}(G)$, we have

$$\sum_{p=1}^{N} \sum_{q=1}^{N} (L_x^{(p)*} L_x^{(p)} \varphi(x) \cdot L_y^{(q)*} L_y^{(q)} \psi(y), f(x, y))_{L^2(G \times G)}$$

$$= \sum_{p=1}^{N} \sum_{q=1}^{N} (L_x^{(p)*} L_x^{(p)} \varphi(x) \cdot L_y^{(q)*} L_y^{(q)} \psi(y), f_0(x, y))_{L^2(G \times G)} .$$

This equation gives us further

$$(A\varphi(x) \cdot A\psi(y), f(x, y) - f_0(x, y))_{L^2(G \times G)} = 0.$$

Setting $u(x) = (f(x, \cdot) - f_0(x, \cdot), A\psi(\cdot))_0$, $x \in G$, we obtain $u \in H_0^m(G)$ and $(u, A\varphi)_0 = 0$. Then we see, by the uniqueness theorem of the Dirichlet problem, that $u = 0$ on G. Therefore it holds, for each $x \in G$, $f(x, \cdot) - f_0(x, \cdot) \in H_0^m(G)$ and

$$(f(x, \cdot) - f_0(x, \cdot), A\psi(\cdot))_0 = 0,$$

from which we obtain $f = f_0 \in H^{\otimes}(A)$. The proof is thus complete.

5. Proofs of Theorems

First, we shall state a condition for two Gaussian measures to be equivalent in terms of the corresponding covariance functions. Let P_ν be the probability measure in $\mathfrak{P}_m$ determined by an operator $A^{(\nu)} \in \mathfrak{A}_m$ ($\nu = 1, 2$). We denote by R_ν and $R_\nu(x, y)$ the Green operator and the Green's function respectively corresponding to $A^{(\nu)}$ ($\nu = 1, 2$). Then we see by the general result [5] that $P_1 \sim P_2$ if and only if

$$R_1(x, y) - R_2(x, y) \in H^{\otimes}(A^{(1)}).$$

Combining this with Lemma 5, we obtain the following

Proposition 1. *Let $A^{(1)}$ and $A^{(2)}$ be in $\mathfrak{A}_m^1$ and $\mathfrak{A}_m$, respectively, where $A^{(1)}$ is represented in the form* (2.1). *The measures P_1 and P_2 are equivalent if and only if*

$$\sum_{p=1}^{N} \sum_{q=1}^{N} \int_G \int_G | L_x^{(p)} L_y^{(q)} [R_1(x, y) - R_2(x, y)]|^2 \, dx \, dy < \infty. \tag{5.1}$$

In this connection we note that the above proposition is similar to the one for the case of homogeneous Gaussian random fields ([7]).

Proof of Theorem 1. Suppose $A^{(1)}$ is represented in the form (2.1). Set $r = \text{order}(A^{(2)} - A^{(1)})$ and

$$u(x) = [(R_1 - R_2)f](x) = \int_G [R_1(x, y) - R_2(x, y)] f(y)\, dy,$$

where f is an arbitrary function in $L^2(G)$. Then the function u is the solution of the Dirichlet problem in the following sense:

$$u \in H_0{}^m(G) \qquad \text{and} \qquad A^{(1)}u = (A^{(2)} - A^{(1)})\, R_2 f.$$

Since $R_2 f \in H^{2m}(G)$ and $(A^{(2)} - A^{(1)})\, R_2 f \in H^{2m-r}(G)$, we obtain $u \in H^{(2m-r)+2m}(G)$ by the regularity theorem. Therefore the image of $L^2(G)$ under $R_1 - R_2$ is contained in $H^{(2m-r)+2m}(G)$. We see now by the kernel theorem that

$$R_1(x, y) - R_2(x, y) \in H^{(2m-r)+2m-[d/2]-1}(G \times G).$$

If $r \leqslant 2m - [d/2] - 1$, we obtain $R_1(x, y) - R_2(x, y) \in H^{2m}(G \times G)$, which implies the equivalence of P_1 and P_2 by Proposition 1. The proof is thus complete.

Proof of Theorem 2. Since $A^{(1)} \in \mathfrak{A}_m{}^2 \subset \mathfrak{A}_m{}^1$, the "if" part follows immediately from Theorem 1. We have now only to prove the "only if" part. Let us prepare some notations used below. We can assume that $A^{(\nu)}$ is represented in the form

$$A^{(\nu)} = \sum_{k=0}^{m} C_k^{(\nu)}(-\Delta)^{m-k},$$

where $C_k^{(\nu)}$'s are nonnegative constants and $C_0^{(\nu)} > 0$ ($\nu = 1, 2$), and that $A^{(1)}$ is also represented in the form (2.1). We denote by $Q_\nu(\xi)$ a polynomial in $\xi_1, \ldots, \xi_d$, given by

$$Q_\nu(\xi) = \sum_{k=0}^{m} C_k^{(\nu)} \left(-\sum_{j=1}^{d} \xi_j{}^2\right)^{m-k}, \qquad \nu = 1, 2.$$

By a fundamental solution of $A^{(\nu)}$ we mean a distribution E_ν in $\mathbb{R}^d$ such that $A^{(\nu)}E_\nu = \delta$ ($\nu = 1, 2$). We can construct these fundamental solutions with the aid of Fourier transforms and show that they are continuous functions on $\mathbb{R}^d$. We remark that functions $H_\nu(x, y)$, $\nu = 1, 2$, on $G \times G$ given by

$$H_\nu(x, y) = E_\nu(x - y) - R_\nu(x, y), \qquad \nu = 1, 2, \tag{5.2}$$

satisfy the conditions

$$H_\nu(x, y) \in \bigcap_{n=0}^{\infty} H^n(G \times G), \qquad \nu = 1, 2. \tag{5.3}$$

In fact, it is easy to see that $A^{(\nu)} H_\nu f = 0$ ($\nu = 1, 2$), where f is an arbitrary function in $L^2(G)$ and H_ν is a continuous linear operator from $L^2(G)$ to $L^2(G)$ given by

$$(H_\nu f)(x) = \int_G H_\nu(x, y) f(y)\, dy, \qquad \nu = 1, 2.$$

We see then by the theorem of Friedrichs ([4]) that $H_\nu f \in \bigcap_{n=0}^{\infty} H^n(G)$ ($\nu = 1, 2$). Therefore we obtain (5.3) by the kernel theorem.

We are now ready to prove the only if part of Theorem 2. Suppose that $P_1 \sim P_2$. Then we have

$$\int_U |(A^{(2)} - A^{(1)})\, E_2(x)|^2\, dx < \infty, \qquad U = \{x \in \mathbb{R}^d;\ |x| < 1\}. \tag{5.4}$$

In fact, it is easy to see by (5.1)–(5.3) that

$$\sum_{p=1}^{N} \sum_{q=1}^{N} \int_G \int_G |L_x^{(p)} L_y^{(q)} [E_1(x - y) - E_2(x - y)]|^2\, dx\, dy < \infty.$$

In view of the representation (2.1) of $A^{(1)}$, where the coefficients of the operators $L^{(p)}$ are constants, it follows that

$$\int_G \int_G |A_x^{(1)} [E_1(x - y) - E_2(x - y)]|^2\, dx\, dy < \infty.$$

Therefore we obtain (5.4) by using the following equation in the sense of distributions:

$$A_x^{(1)} [E_1(x - y) - E_2(x - y)] = (A_x^{(2)} - A_x^{(1)})\, E_2(x - y).$$

In order to replace the condition (5.4) by the one concerning the polynomials $Q_\nu(\xi)$ ($\nu = 1, 2$), we construct the fundamental solution $E_2(x)$ of $A^{(2)}$ by using the technique of Hörmander ([6]). Then $E_2(x)$ can be written in the following form:

$$E_2(x) = F_{20}(x) + F_2(x), \tag{5.5}$$

where

$$F_{20}(x) = \mathfrak{F}\left[\frac{1-\chi_T(\xi)}{Q_2(i\xi)}\right](x) \text{ symbolically,}^1$$

$$F_2(x) = \mathfrak{F}\left[\frac{\chi_T(\xi)}{Q_2(i\xi)}\right](x),$$

$$\begin{aligned} \chi_T(\xi) &= 1, \quad & |\xi| \geqslant T, \\ &= 0, \quad & |\xi| < T, \end{aligned} \tag{5.6}$$

and T is some positive constant. We remark that $F_{20}(x)$ is infinitely differentiable on $\mathbb{R}^d$ and $F_2(x)$ is infinitely differentiable on $\mathbb{R}^d - \{0\}$. Therefore we see by (5.4) and (5.5) that

$$\int_U |(A^{(2)} - A^{(1)})\, F_2(x)|^2\, dx < \infty. \tag{5.7}$$

Set $r = \operatorname{order}(A^{(2)} - A^{(1)})$ and

$$M(x) = \sum_{k=0}^{m} (C_k^{(2)} - C_k^{(1)})(-1)^{m-k}\, \mathfrak{F}\left(\chi_T(\xi)\, \frac{(\sum_{j=1}^d i^2\xi_j^2)^{m-k}}{Q_2(i\xi)}\right)(x). \tag{5.8}$$

Then we see by (5.6)–(5.8) that $M(x) \in L^2(U)$. Thus it suffices to show $r \leqslant 2m - [d/2] - 1$ under the condition $M(x) \in L^2(U)$. Set

$$S_1(\xi) = C_0^{(2)}\left(-\sum_{j=1}^{d} \xi_j^2\right)^m \quad \text{and} \quad S_2(\xi) = \sum_{k=1}^{m} C_k^{(2)}\left(-\sum_{j=1}^{d} \xi_j^2\right)^{m-k}.$$

By using the relation

$$\frac{1}{Q_2(i\xi)} = \frac{1}{S_1(i\xi)} - \frac{S_2(i\xi)}{S_1(i\xi)} \cdot \frac{1}{Q_2(i\xi)}, \qquad |\xi| \geqslant T,$$

repeatedly, we obtain the following expansion:

$$\frac{(\sum_{j=1}^d i^2\xi_j^2)^{m-k}}{Q_2(i\xi)} = \sum_{l=0}^{mn_0} C(k,l) \cdot |\xi|^{-(2k+2l)} + \rho_{k,n_0}(\xi), \quad |\xi| \geqslant T, \tag{5.9}$$

[1] $\mathfrak{F}f$ denotes the inverse Fourier transform of f defined by

$$(\mathfrak{F}f)(x) = (1/(2\pi)^d) \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\, d\xi.$$

where $C(k, l)$'s are constants, $C(k, 0) \neq 0$, and n_0 is a positive integer such that $\rho_{k,n_0}(\xi) = o(|\xi|^{-d-1})$ as $|\xi| \to \infty$ $(0 \leqslant k \leqslant m)$. Set

$$K_p(x) = \overline{\mathfrak{F}}[\chi_T(\xi) \cdot |\xi|^{-2p}](x), \qquad p \geqslant 0, \tag{5.10}$$

and

$$R_k(x) = \overline{\mathfrak{F}}[\chi_T(\xi) \cdot \rho_{k,n_0}(\xi)](x), \qquad 0 \leqslant k \leqslant m. \tag{5.11}$$

Then we see by (5.8)–(5.11) that

$$\begin{aligned} M(x) &= (C_0^{(2)} - C_0^{(1)})(-1)^m C(0,0) K_0(x) \\ &\quad + (C_0^{(2)} - C_0^{(1)})(-1)^m \sum_{l=1}^{mn_0} C(0, l) K_l(x) \\ &\quad + \sum_{k=1}^{m} \sum_{l=0}^{mn_0} (C_k^{(2)} - C_k^{(1)})(-1)^{m-k} C(k, l) K_{k+l}(x) \\ &\quad + \sum_{k=0}^{m} (C_k^{(2)} - C_k^{(1)})(-1)^{m-k} R_k(x). \end{aligned} \tag{5.12}$$

We can show that $R_k(x)$, $0 \leqslant k \leqslant m$, are bounded continuous functions on $\mathbb{R}^d$ and $K_p(x)$, $0 \leqslant p \leqslant (d-1)/2$, are written in the form

$$K_0(x) = \delta(x) + J_0(x), \tag{5.13}$$

$$K_p(x) = C_{p,d} \cdot |x|^{2p-d} + J_p(x), \qquad 1 \leqslant p \leqslant (d-1)/2, \tag{5.14}$$

where $J_p(x)$, $0 \leqslant p \leqslant (d-1)/2$, are bounded continuous functions on $\mathbb{R}^d$ and $C_{p,d}$, $1 \leqslant p \leqslant (d-1)/2$, are nonzero constants. Further we can prove the following inequalities in U:

$$\begin{aligned} |K_p(x)| &\leqslant C_0 \cdot |x|^{2p-d}, & 1 \leqslant p \leqslant (d-1)/2, \\ &\leqslant C_0 \cdot (\log(1/|x|) + 1), & p = d/2, \\ &\leqslant C_0, & p \geqslant (d+1)/2, \end{aligned} \tag{5.15}$$

where C_0 is some positive constant. Since $M(x) \in L^2(U) \subset L^1(U)$, $R_k(x) \in L^1(U)$ $(0 \leqslant k \leqslant m)$, $K_p(x) \in L^1(U)$ $(p \geqslant 1)$ and $K_0(x) \notin L^1(U)$, we see by (5.12) that $C_0^{(1)} = C_0^{(2)}$. Therefore $M(x)$ can be written in the form

$$\begin{aligned} M(x) &= (C_{m-r/2}^{(2)} - C_{m-r/2}^{(1)})(-1)^{r/2} C(m - (r/2), 0) K_{m-r/2}(x) \\ &\quad + \sum_{p=1}^{n_1} \lambda_p \cdot K_{m-r/2+p}(x) + R(x), \qquad n_1 = mn_0 + (r/2), \end{aligned} \tag{5.16}$$

where λ_p, $1 \leqslant p \leqslant n_1$, are constants and $R(x)$ is a bounded continuous function on $\mathbb{R}^d$.

Now, we have only to consider two cases: (1) $m - (r/2) > (d - 1)/2$; (2) $m - (r/2) \leqslant (d - 1)/2$. In case (1), we directly obtain $r \leqslant 2m - d \leqslant 2m - [d/2] - 1$. In case (2), we see by (5.14)–(5.16) that

$$M(x) = (C^{(2)}_{m-r/2} - C^{(1)}_{m-r/2})(-1)^{r/2}\, C(m - (r/2), 0)\, C_{m-r/2,d} \cdot |\, x\,|^{2m-r-d}$$
$$+ \sum_{p=1}^{n_2} \mu_p \cdot |\, x\,|^{2m-r-d+2p} + J(x), \quad n_2 = [(d-1)/2] - (m - (r/2)), \tag{5.17}$$

where μ_p, $1 \leqslant p \leqslant n_2$, are constants and $J(x)$ is a continuous function on $U - \{0\}$ satisfying the following inequality in U:

$$|\, J(x)| \leqslant C_0' \cdot (\log(1/|\, x\,|) + 1), \qquad C_0' > 0.$$

Therefore we can show the following inequality in some neighborhood U_0 of $x = 0$, $U_0 \subset U$:

$$|\, M(x)| \geqslant C \cdot |\, x\,|^{2m-r-d}, \qquad C > 0.$$

Then we see by $M(x) \in L^2(U_0)$ that $|\, x\,|^{2m-r-d} \in L^2(U_0)$, which implies that $r \leqslant 2m - [d/2] - 1$. The proof is thus complete.

References

[1] Agmon, S. (1965). *Lectures on Elliptic Boundary Value Problems*. Van Nostrand, Princeton, New Jersey.

[2] Aronszajn, N. (1950). Theory of reproducing kernels. *Trans. Amer. Math. Soc.* **68** 337–404.

[3] Fernique, X. (1964). Continuité des processus Gaussiens. *C. R. Acad. Sci. Paris Ser. A* **258** 6058–6060.

[4] Friedrichs, K. (1953). On the differentiability of the solutions of linear elliptic differential equations. *Comm. Pure Appl. Math.* **6** 299–325.

[5] Golosov, Ju. I. and Tempel'man, A. A. (1969). On equivalence of measures corresponding to Gaussian vector-valued functions. *Soviet Math. Dokl.* **10** 228–232.

[6] Hörmander, L. (1955). On the theory of general partial differential operators. *Acta Math.* **94** 161–248.

[7] Skorokhod, A. V. and Yadrenko, M. I. (1973). On absolute continuity of measures corresponding to homogeneous Gaussian fields. *Theory Probab. Its Appl. USSR* **18** 27–40.